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ON THE CENTRE-FOCUS PROBLEM IN SOME LIÉNARD SYSTEMS

O PROBLEMU CENTRA IN FOKUSA V NEKATERIH LIÉNARDOVIH SISTEMIH

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Keywords: Polynomial system of ODEs, Center-focus problem, Complex behavior

Abstract

A large family of planar systems of ODEs arising from Liénard equations is considered. A Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ is commonly used in practical problems, in particular in (electro)mechanics. It is well-known that a Liénard equation can be transformed into an autonomous planar system of ODEs of the form $x' = y - F(x)$, $y' = -g(x)$, where $F'(x) = f(x)$. In this paper $f(x) = 2a_2x + 3a_3x^2 + 4a_4x^3$ and $g(x) = c_1x + c_3x^3 + c_5x^5 + c_7x^7$. In the parameter space $(a_2, a_3, a_4, c_1, c_3, c_5, c_7) \in \mathbb{R}^7$ we consider the center-focus problem and find necessary conditions for the corresponding system having a center at the origin. In the parameter space $(a_2, a_3, a_4, c_1, c_3, c_5, c_7) \in \mathbb{R}^7$ an example with a possible limit cycle and some examples with other complex dynamic behavior are presented.

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Povzetek

Obravnavana je velika družina ravninskih sistemov navadnih diferencialnih enačb (NDE), ki izhaja iz Liénardove enačbe. Liénardova enačba $\ddot{x}+f(x)\dot{x}+g(x)=0$ se uporablja v praktičnih problemih, še zlasti v (elektro)mehaniki. Znano je, da se Liénardova enačba lahko preoblikuje v avtonomni ravninski sistem NDE oblike $x'=y-F(x)$, $y'=-g(x)$, kjer je $F'(x)=f(x)$. V tem članku sta $f(x)=2a_2x+3a_3x^2+4a_4x^3$ in $g(x)=c_1x+c_3x^3+c_5x^5+c_7x^7$. V prostoru parametrov $(a_2,a_3,a_4,c_1,c_3,c_5,c_7)\in\mathbb{R}^7$ obravnavamo problem centra in fokusa in poiščemo potrebne pogoje, da ima ustrezen sistem center v izhodišču. Prav tako v prostoru parametrov $(a_2,a_3,a_4,c_1,c_3,c_5,c_7)\in\mathbb{R}^7$ predstavimo primer z možnim limitnim ciklom in nekatere primere z drugimi kompleksnimi dinamičnimi pojavi.

1 INTRODUCTION

A great number of mathematical models of physical systems give rise to differential equations of the type

$$\ddot{x}+f(x)\dot{x}+g(x)=0 \tag{1.1}$$

which is called a Liénard equation. From the mechanical point of view, equation (1.1) can be interpreted as the generalization of the mass-spring-damper system, where $f(x)\dot{x}$ is the damping term and $g(x)$ represents the (nonlinear) spring term. Applications of equation (1.1) can be found in many important examples including chemical reactions, predator-prey models, vibration analysis, etc.

Two famous examples of a Liénard equation are the Van der Pol equation and Duffing's equation. The Van der Pol equation

$$\ddot{x}+\varepsilon(x^2-1)\dot{x}+x=0, \quad \varepsilon>0$$

describes the circuit of a vacuum tube, whilst Duffing's equation

$$\ddot{x}+\delta\dot{x}+\alpha x+\beta x^3=\gamma\cos(\omega t),$$

aims to model certain nonlinearly damped/driven oscillators (i.e. a spring pendulum whose spring's stiffness does not exactly obey Hooke's law). Here $x=x(t)$ represents the displacement of the (pendulum) bob at time t, $\dot{x}$ represents the first derivative of x with respect to time t, and $\ddot{x}$ is the second time-derivative of x. Parameters $\alpha,\beta,\gamma,\delta$ and $\omega>0$ are given (real) constants (case $\beta=\delta=0$ corresponds to simple harmonic motion).

There are two conventional transitions from homogeneous ODE (1.1) to a planar dynamical system of ODEs. Namely, setting $y=\dot{x}$ we obtain

$$x'=y, \qquad y'=-f(x)y-g(x). \tag{1.2}$$

In (1.1) another approach is possible via the so-called Liénard coordinates. Substituting $y=\dot{x}+F(x)$, where $f(x)=F'(x)$, one obtains

$$x' = y - F(x),\ y' = -g(x). \tag{1.3}$$

Both systems (1.2) and (1.3) are special cases of (continuous) dynamic system called autonomous systems of ODEs, which generally takes the form

$$x' = P(x, y), \qquad y' = Q(x, y). \tag{1.4}$$

Even in planar dynamics, there are several open problems: among them the problem of finding the position and the number of limit cycles bifurcating from a non-hyperbolic singular point, which is a part of famous Hilbert's 16th problem. Most real-life problems are related to the centre-focus problem: this is of distinguishing between a centre, where all orbits in the neighbourhood of the singular point are periodic, and a focus, where all orbits are spiralling away or towards to the singular point/origin (for more details see e.g. *[1]*). Autonomous systems (1.4) cannot contain chaotic dynamics (according to the Poincaré-Bendixon theorem). However, note that Duffing's equation (which is a nonhomogeneous ODE of order two) corresponding to a non-autonomous system $x' = \tilde{P}(x, y, t),\ y' = \tilde{Q}(x, y, t)$) is an example of a dynamical system that exhibits chaotic behaviour. In contrast Duffing's equation is also a classic example of a dynamical system with a limit cycle, *[2,3]*. For example, for $f(x) = \varepsilon(x^2 - 1)$ and $g(x) = x$ we obtain

$$x' = y - \varepsilon\left(\frac{x^3}{3} - x\right), \qquad y' = -x$$

which readily contains periodic solutions for $x(t)$ and $y(t)$. In Figs.1-3, there is an example for $\varepsilon = 1.2$ with initial conditions $x(0) = -0.4$, $y(0) = 0.3$. In Fig. 4, the corresponding stream-line plot with a clearly visible limit cycle is presented.

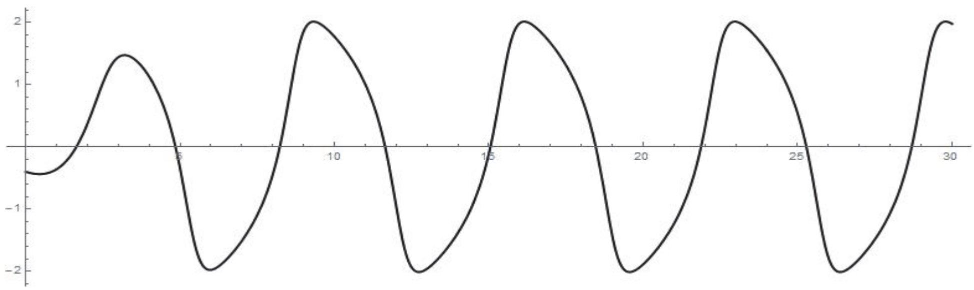

Figure 1: *Initial value solution* $x = x(t)$ *to Duffing's equation for* $\varepsilon = 1.2$*;* $x(0) = -0.4$*,* $y(0) =$*0.3*

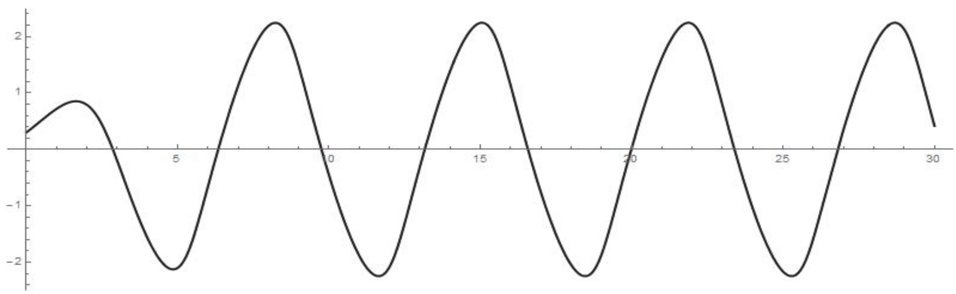

Figure 2: *Initial value solution* $y = y(t)$ *to Duffing's equation:* $\varepsilon = 1.2$; $x(0) = -0.4$, $y(0) = 0.3$

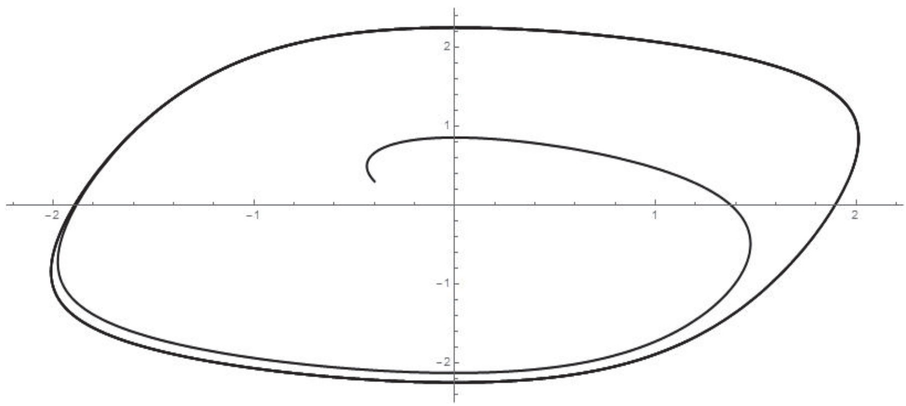

Figure 3: *A solution tending to limit cycle of Duffing's equation:* $\varepsilon = 1.2$ *and* $x(0) = -0.4$, $y(0) = 0.3$

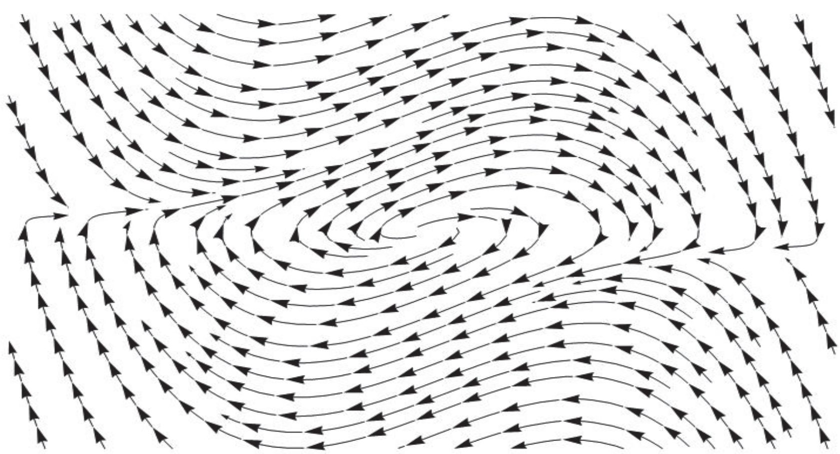

Figure 4: *Streamline-plot (Mathematica) of Duffing's equation for* $\varepsilon = 1.2$ *containing a limit cycle*

Usually for systems (1.4) and, consequently, also for systems (1.2) and (1.3) singular points $\vec{x}_0$ are hyperbolic; this means that no eigenvalue of the Jacobian matrix

$$J(\vec{x}_0) = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix}, \text{ where } \vec{x} = (x, y)$$

evaluated at $\vec{x}_0$ has the real part equal to zero. In this case the linearized system $\vec{x}' = J(\vec{x}_0)$ may (locally – near singular point $\vec{x}_0 = (x_0, y_0)$) be used to approximate the original system (1.4). More precisely, the approximation is in terms of a continuous invertible map, which locally (near singularity) takes parametrized solutions of the linearized system $\vec{x}' = J(\vec{x}_0)$ to the parametrized solutions of the original system (1.4). This is the statement of the Hartman—Grobman theorem *[4]*. For any singular point $\vec{x}_0$ of system (1.4) one of the following five »generic cases«, according to the Jordan canonical form of $J(\vec{x}_0)$, appears

$$(i)\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \lambda_{1,2} \neq 0 \ \ (ii)\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, (iii)\begin{bmatrix} \omega & 0 \\ 0 & 0 \end{bmatrix}, (iv)\begin{bmatrix} 0 & \omega \\ 0 & 0 \end{bmatrix}, \omega \neq 0 \ \ (v)\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

None, except the first one, are within the scope of the Hartman—Grobman theorem (for any $\omega \neq 0$) and must be considered case by case/separately. Systems corresponding to the Jacobian matrix of the form (ii) are the most studied planar systems. The addition of nonlinear terms may result either in centre or in focus. In this paper, we will consider a special case of (1.2) of the form

$$x' = y, \qquad y' = -(2a_2x + 3a_3x^2 + 4a_4x^3)y - (c_1x + c_3x^3 + c_5x^5 + c_7x^7), \qquad (1.5)$$

where $a_2, a_3, a_4, c_1, c_3, c_5, c_7 \in \mathbb{R}$, and analyse the stability for the whole family (1.5).

2 THE ANALYSIS OF SYSTEM (1.5)

2.1 Case $c_1 > 0$

If $c_1 > 0$ introducing new coordinates $X = c_1x, Y = c_1y$ one can obtain system $X' = Y$, $Y' = -(2a_2X + 3a_3X^2 + 4a_4X^3)Y - (X + c_3X^3 + c_5X^5 + c_7X^7)$. The corresponding Jacobian at singular point (0,0) is

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which yields a centre or focus at the origin. In Fig. 5 there is a stream-line plot of a centre.

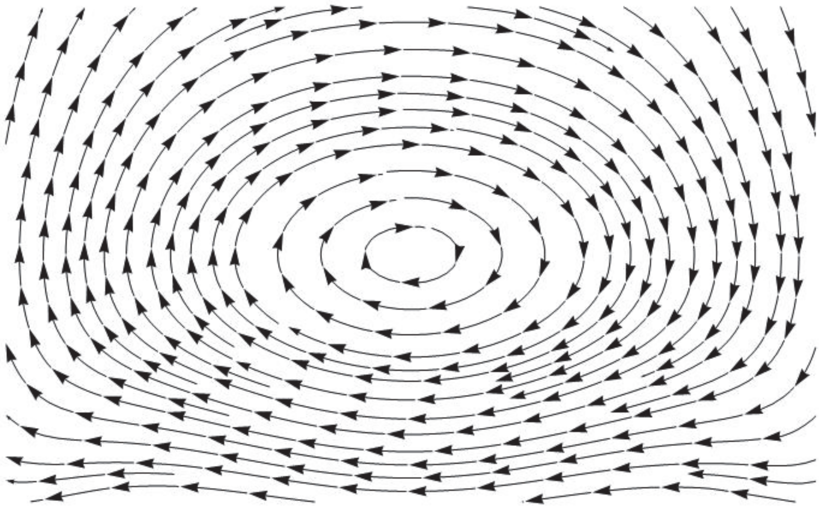

Figure 5: *Centre for system (5), with* $a_2 = a_3 = 0,\ a_4 = \frac{3}{4}, c_1 = 1, c_3 = c_5 = 0, c_7 = 2$

Note that the complete analysis of case $c_1 > 0$ is done in a separate section in the sequel; it will be seen that $a_3 = 0$ is the sufficient condition to obtain a centre in system (1.5). However, distinguishing between a centre and a focus just from a streamline plot (e.g. in Mathematica) is impossible and much too inaccurate (since the stream-plot of a focus is too similar to the stream-plot of a centre). In the case of a focus, it is much more convenient to consider a single solution, like in Figs. 6-7 in which graphs of $x = x(t)$ and $y = y(t)$ are shown. In Fig. 8, the parametric solution $(x(t), y(t))$ is shown in the phase plane (x, y).

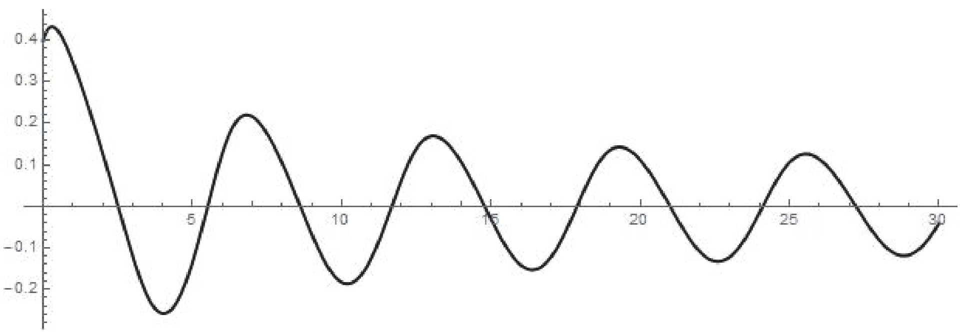

Figure 6: *Particular solution* $x = x(t)$ *of (1.5) with* $a_2 = a_4 = 0,\ a_3 = c_1 = 1, c_3 = 2, c_5 = c_7 = 0$

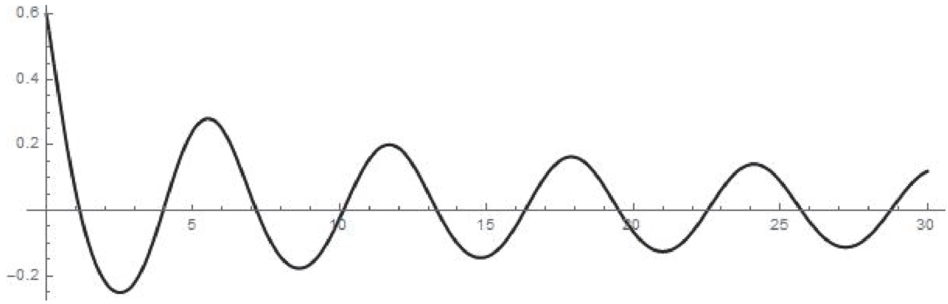

Figure 7: *Particular solution* $y = y(t)$ *of (1.5) with* $a_2 = a_4 = 0,\ a_3 = c_1 = 1, c_3 = 2, c_5 = c_7 = 0$

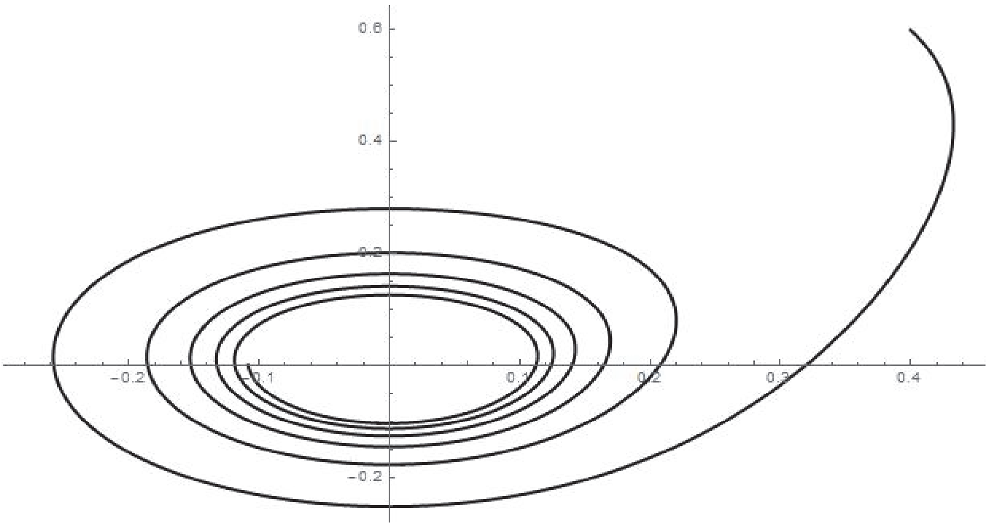

Figure 8: Trajectory $(x(t), y(t))$ of (1.5) with $a_2 = a_4 = 0,\ a_3 = 1, c_1 = 1, c_3 = 2, c_5 = c_7 = 0$

2.2 Case $c_1 < 0$

If $c_1 < 0$ introducing new coordinates $X = c_1 x, Y = c_1 y$ one yields $X' = Y,\ Y' = -(2a_2 X + 3a_3 X^2 + 4a_4 X^3)Y - (-X + c_3 X^3 + c_5 X^5 + c_7 X^7)$. The corresponding Jacobian matrix at a singular point (0,0) is

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of $J(0,0)$ are $\lambda_{1,2} = \pm 1$, yielding a hyperbolic singular point: a saddle with an unstable singularity. The dynamics near the origin (according to the Hartman–Grobman theorem) are topologically conjugate to the linear system $X' = Y,\ Y' = X$ (see Fig. 9), and no further analysis is needed in this case.

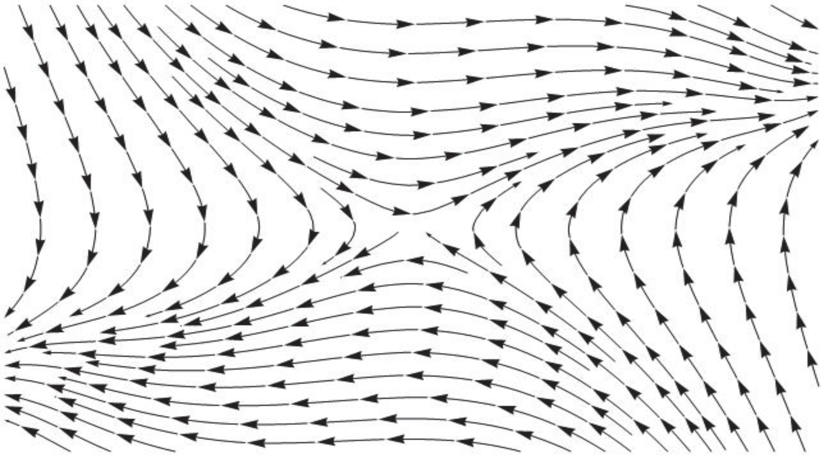

Figure 9: *Phase portrait of a saddle for (1.5);* $a_2 = a_4 = 0,\ a_3 = 1, c_1 = -1, c_3 = c_5 = 0, c_7 = 1$

2.3 Case $c_1 = 0$

If $c_1 = 0$, the system takes the form $x' = y$, $y' = -(2a_2x + 3a_3x^2 + 4a_4x^3)y - (c_3x^3 + c_5x^5 + c_7x^7)$, and $(0,0)$ is a non-elementary (nilpotent) singular point if type (iv), since the corresponding Jacobian at singular point $(0,0)$ is

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The blow-up desingularization gives rise to the type of stability in this case (see *[5,6]* for details). Note that the case (v) can never appear for Lienard systems since $\frac{\partial P}{\partial y} = 1 \neq 0$.

3 THE CENTRE-FOCUS ANALYSIS

In order to consider the centre-focus problem, we briefly recall an approach *[1]* for studying the problem for planar polynomial systems (1.4), where $P(x,y) = y + h.o.t.$ and $Q(x,y) = -x + h.o.t.$ Here $h.o.t.$ stands for $\sum_{i+j=2}^{n} P_{i,j}x^iy^j$, and $\sum_{i+j=2}^{n} Q_{i,j}x^iy^j$, respectively. Note that via the (reverse) time transformation $\tau = -t$ this is equivalent to $P(x,y) = y + h.o.t.$ and $Q(x,y) = -x + h.o.t.$, therefore we make no difference between these two cases. Here $P(x,y)$ and $Q(x,y)$ are polynomials of degree at most n without constant and linear terms. It is convenient to introduce the polar coordinates $u = r\cos\varphi$, $v = r\sin\varphi$ and consider the so-called Poincaré return map $R(r)$. Introducing the polar coordinates into (1.4) with $P(x,y) = y + h.o.t.$ and $Q(x,y) = -x + h.o.t.$ yields the equation of the trajectories

$$\frac{dr}{d\varphi} = \frac{r^2F(r,\cos\varphi,\sin\varphi)}{1 + rG(r,,\cos\varphi,\sin\varphi)} = R(r,\varphi).$$

Obviously, $r=0$ is a solution to $\frac{dr}{d\varphi}=R(r,\varphi)$, since $R(0,\varphi)\equiv 0$. The function $R(r,\varphi)$ is periodic (with the least period 2π in variable φ) and analytic for (small enough) $|r|\le r^*$ (and all φ). Thus, $R(r,\varphi)$ can be expanded in a convergent power series in r to obtain

$$\frac{dr}{d\varphi}=\mathrm{r}^2R_2(\varphi)+\mathrm{r}^3R_3(\varphi)+\cdots \tag{3.1}$$

The continuous dependence on the initial conditions and the fact that $r=0$ is a solution for all $\varphi\in[0,2\pi]$ yield that every solution to the above equation in a sufficiently small neighbourhood of the origin intersects every ray φ=φ_0, $\varphi_0\in[0,2\pi]$. This means that without loss of generality one can choose the line segment $\Sigma=\{(u,v);\ v=0,0\le u\le r^*\}$, where r^* is chosen to be small enough.

Next, we consider the solution of (3.1) with the initial condition $r(\varphi=0)=r_0$ and expand it into a power series in r_0 to obtain

$$r(\varphi,r_0)=w_1(\varphi)r_0+w_2(\varphi)r_0^2+w_3(\varphi)r_0^3+\cdots \tag{3.2}$$

which is also convergent for all $\varphi\in[0,2\pi]$ and all $|r_0|\le\ r^*$. The $r(\varphi)$ from (3.2) is a solution of (3.1) and inserting $r(\varphi,r_0)$ into (3.2) yields recurrence differential equations for functions $w_j(\varphi)$ (see *[1]* for more details). We consider one revolution of $r=r(\varphi,r_0)$ beginning on $r_0\in\Sigma$ where φ is assumed to be 0 and study the return to Σ (which occurs at $\varphi=2\pi$, that is, after one revolution). Thus, the Poincaré return map $R(r)$ is defined by

$$\mathcal{R}(r_0)=r(2\pi,r_0)=r_0+w_2(2\pi)r_0^2+w_3(2\pi)r_0^3+\cdots$$

The coefficients $\eta_j:=w_j(2\pi)$ for $j>1$, defined in the above equation are called *Lyapunov numbers.*

From the definition of polar coordinates, we readily conclude that the zeros of the difference function $\mathcal{R}(r_0)-r_0$ correspond to closed orbits. In particular, isolated zeros correspond to limit cycles, and if $\mathcal{R}(r_0)-r_0\equiv 0$, the system has a centre at the origin, which means that for all $j>1$ the Lyapunov numbers η_j must vanish.

However, computing Lyapunov numbers requires the integration of trigonometric functions, which can be very difficult problems for some cases. Poincare and Lyapunov proved that system (1.4) with $P(x,y)=y+h.o.t.$ and $Q(x,y)=-x+h.o.t.$ has centre at the origin if it admits the first integral of the form

$$\Phi(x,y)=x^2+y^2+\sum_{i+j\ge 3}v_{ij}\,x^iy^j, \tag{3.3}$$

which is an analytic function in a neighbourhood of the origin $(0,0)$. By definition, Φ is a first integral of system (1.4) if Φ is a solution to the following PDE

$$\Psi:=\frac{\partial\Phi(x,y)}{\partial x}\cdot P(x,y)+\frac{\partial\Phi(x,y)}{\partial y}Q(x,y)\equiv 0. \tag{3.4}$$

In agreement with formula (3.4), equation $\Psi\equiv 0$ can be solved only on some special variety (set of zeros) in the (affine) space of parameters a_{ij},b_{ij} defined by coefficients of $P(x,y)$ and $Q(x,y)$. Generally, from (3.4) using step-by-step process of equating the proper coefficients of Ψ to zero we obtain

$$\Psi = g_4(a_{ij}, b_{ij})(x^2 + y^2)^2 + g_6(a_{ij}, b_{ij})(x^2 + y^2)^3 + g_8(a_{ij}, b_{ij})(x^2 + y^2)^4 + \cdots$$

yielding $\Psi \equiv 0$ if and only if

$$g_4(a_{ij}, b_{ij}) = g_6(a_{ij}, b_{ij}) = \cdots = g_{2k}(a_{ij}, b_{ij}) = \cdots = 0 \qquad (3.5)$$

for all $k \geq 2$. The numbers (polynomials) $g_{2k}(a_{ij}, b_{ij})$ are called *focus quantities [1]*. Finally, the centre-focus problem reduces to find the conditions for vanishing all focus quantities $g_{2k}(a_{ij}, b_{ij})$.

For case (1.5), we computed the first several polynomials $g_{2k}(a_{ij}, b_{ij})$ using and obtained

$g_4 = -\frac{3a_3}{4}$,

$g_6 = \frac{a_3}{32}(76a_2^2 + 21c_3)$,

$g_8 = -\frac{a_3}{3072}(33834a_2^4 + 19736a_2^2c_3 + 9(133a_3^2 + 229c_3^2 - 216c_5) - 15728a_2a_4)$,

$g_{10} = \frac{a_3}{184320}(154672244a_2^6 - 11734288a_2^3a_4 + 10684399a_2^4c_3 - 2776152a_2a_4c_3 - 3a_2^2(484752a_3^2 - 789569c_3^2 + 559144c_5) - 3a_2^2(484752a_3^2 - 789569c_3^2 + 559144c_5) + 9(52064a_4^2 + 5(4929a_3^2c_3 + 2901c_3^3 - 6164c_3c_5 + 2568c_7)))$, etc.

The expressions for the other focus quantities are too long to be presented here, but can easily be computed using Mathematica or any other appropriate computer algebra system. Obviously, the condition $a_3 = 0$ is necessary for (3.5), since a_3 is a cofactor of $g_{2k}(a_{ij}, b_{ij})$ for any k.

This means that system (1.5) for $a_3 = 0$ and $c_1 > 0$ has centre at the origin for arbitrary $a_2, a_4, c_3, c_5, c_7 \in \mathbb{R}$ and the phase portrait in the neighbourhood of the origin is topologically equivalent to the phase portrait shown in Fig. 5.

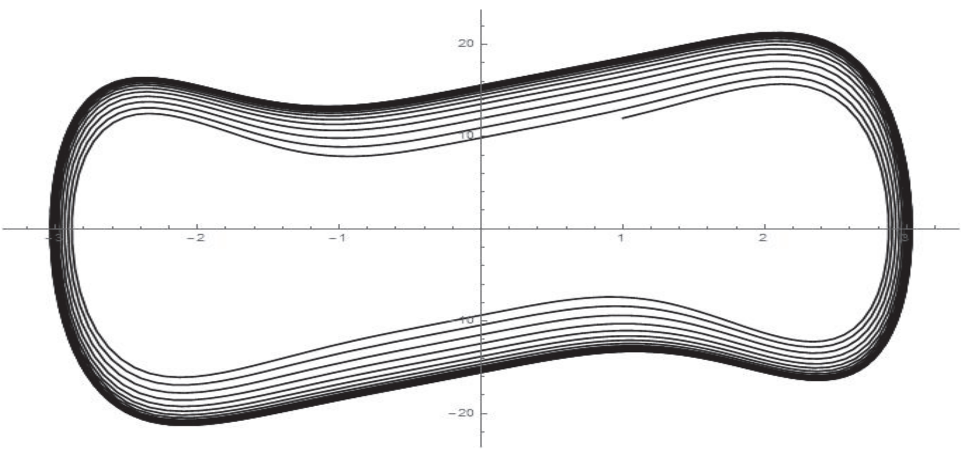

Figure 10: *A non-trivial limit cycle for (1.2) of the form* $x' = y, y' = -(x^2 - 3)y - 5x^3(x^2 - 5)$

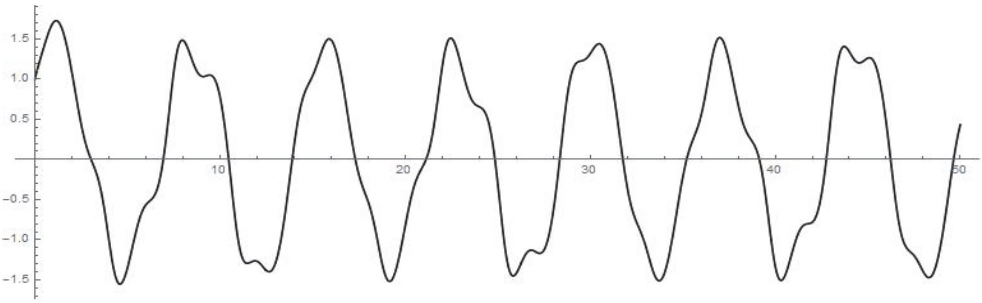

Figure 11: *Complex dynamics of* $x' = y, y' = (\frac{1}{2} - x^2)y - \frac{x^3(x^2-4)}{16} - x + 2\sin 3t$: $x = x(t)$

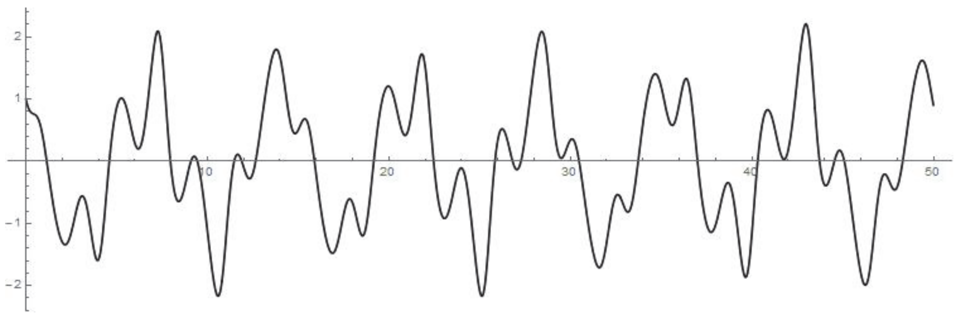

Figure 12: *Complex dynamics of* $x' = y, y' = (\frac{1}{2} - x^2)y - \frac{x^3(x^2-4)}{16} - x + 2\sin 3t$: $y = y(t)$

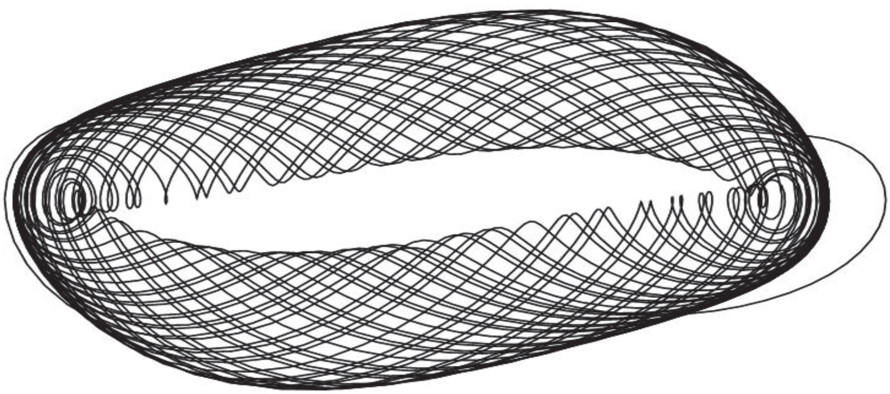

Figure 13: *Complex dynamics of* $x' = y, y' = (\frac{1}{2} - x^2)y - \frac{x^3(x^2-4)}{16} - x + 2\sin 3t$: $(x(t), y(t))$

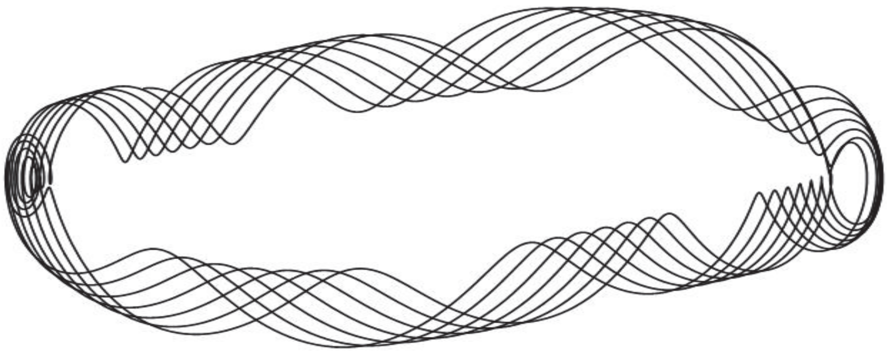

Figure 14: *Complex dynamics of* $x' = y, y' = (\frac{1}{2} - x^2)y - \frac{x^3(x^2-4)}{16} - x - 3\sin 7t$: $(x(t), y(t))$

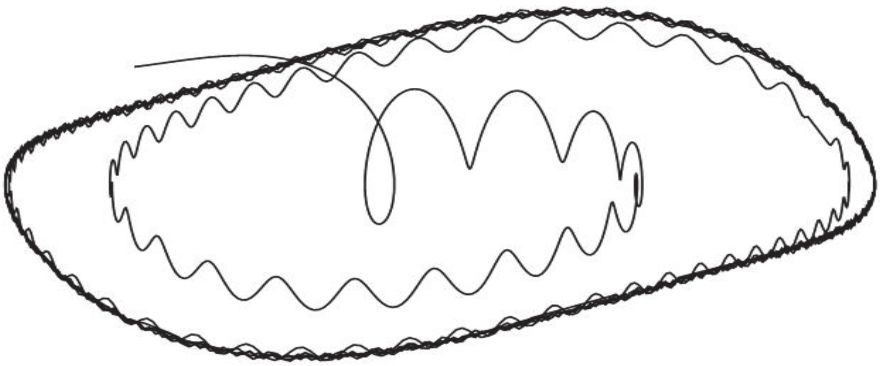

Figure 15: *A single trajectory of* $x' = y, y' = (\frac{1}{2} - x^2)y - \frac{x^3(x^2-4)}{16} - x - 3\sin 2t^2$: $(x(t), y(t))$

4 CONCLUSIONS

The centre-focus problem for system (1.5) is solved for necessary conditions. It is proven that the crucial parameter that distinguishes between the centre and focus in (1.5) is a_3. The condition $a_3 \neq 0$ yields focus, while $a_3 = 0$ defines a centre for any choice of other parameters. Note also that there are many nontrivial examples of system (1.2) that probably admit limit cycles, for example system $x' = y, y' = -(x^2 - 3)y - 5x^3(x^2 - 5))$ shown in Fig.10. According to the well-known Liénard theorem, *[3]*, system (1.2) admits a unique stable limit cycle if f and g are continuously differentiable, g is odd and $g(x)$ >0 for $x > 0$, f is an even function and $\int_0^x f(u)\,du < 0$ for $0 < x < a$, and $\int_0^x f(u)\,du > 0$ and increasing for $x > a$; for some $a \in \mathbb{R}_+$.

Finally, note that a non-autonomous modification of system (1.2): $x' = y,\ y' = -f(x)y - g(x) + h(t)$ may exhibit other complex dynamics, as shown in Figs. 11-15.

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